AD-A114 486 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
A SPECTRAL MAPPING THEOREM FOR THE EXPONENTIAL FUNCTION, AND SO--ETC(U)
JAN 82 T KATO DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2316 NL

1 OF 1
AD
A114486

END
DATE
FILMED
6-82
DTIC

ADA11443

MRC Technical Summary Report #2316

A SPECTRAL MAPPING THEOREM FOR THE EXPONENTIAL FUNCTION, AND SOME COUNTEREXAMPLES

Tosio Kato

Mathematics Research Center
University of Wisconsin–Madison
610 Walnut Street
Madison, Wisconsin 53706

January 1982

(Received October 15, 1981)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

A SPECTRAL MAPPING THEOREM FOR THE EXPONENTIAL FUNCTION AND SOME COUNTEREXAMPLES

Tosio Kato

Technical Summary Report #2316

January 1982

Sigma (e Superscript A)

(Superscript Sigma(A)) C sub O

ABSTRACT

Elementary proofs are given for the (known) theorems that (1) $e^{\sigma(A)} \subset \sigma(e^A)$ if A is the generator of a C_0-semigroup $\{e^{tA}\}$ of linear operators on a Banach space X, and that (2) $e^{\sigma(A)} = \sigma(e^A)\setminus\{0\}$ if $\{e^{tA}\}$ is a holomorphic semigroup. Also a large class of strongly continuous groups $\{e^{tA}\}$ on a Hilbert space H is given such that $\sigma(A)$ is empty. Note that $\sigma(e^A)$ is not empty, and is away from zero, if $\{e^{tA}\}$ is a group. Some related remarks are given on the relationship between the spectral bound of A and the type of $\{e^{tA}\}$.

(e Superscript tA) (Such point of) (belongs to) (e superscript tA) (Sigma)

AMS (MOS) Subject Classifications: 26A33, 47A10, 47A60, 47B25, 47B44, 47B58, 47D05, 47D10.

Key Words: C_0-semigroups, holomorphic semigroups, spectral mapping theorem, spectral radius, spectral bound, type.

Work Unit Number 1: Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

If one solves a system of linear differential equation $dx/dt = Ax$, where $x = x(t)$ is an n-vector and A is an $n \times n$ matrix, the solution may be written $x(t) = e^{tA}x(0)$. Here the matrix e^{tA} may be given by the Taylor series $\Sigma_{k=0}^{\infty} t^k A^k/k!$, or it may be more easily computed if the Jordan canonical form of A is known. In any case if $\lambda_1,\ldots,\lambda_n$ (some of which may be equal) are the eigenvalues of A, then $e^{\lambda_1 t},\ldots,e^{\lambda_n t}$ are the eigenvalues of e^{tA} (this is a special case of the so-called spectral mapping theorem).

It follows that the growth rate of $x(t)$ in any norm is at most exponential: $\|x(t)\| \leq Me^{\omega t}\|x(0)\|$ for $t \geq 0$. The infimum of all possible ω is called the type of the semigroup $\{e^{tA}\}$, and will be denoted by type A in the sequel. The spectral mapping theorem mentioned above implies that $\omega = \max\{\mathrm{Re}\, \lambda_i\}$, which is called the spectral bound of A and will be denoted by spb A. Thus one has the relation spb A = type A.

If the matrix A is replaced with a linear operator A in an infinite-dimensional Banach space X, one can still solve $dx/dt = Ax$ for $x = x(t) \in X$ under certain conditions on A, to obtain a unique solution $x(t) = e^{tA}x(0)$ for $t \geq 0$. A is called the generator of the semigroup $\{e^{tA}\}$. Many problems in linear partial differential equations can be covered by semigroup theory. Here again one can define type A via the optimal growth rate for $\|x(t)\|/\|x(0)\|$, and spb A using the spectrum $\sigma(A)$ of A rather than the eigenvalues. It turns out, however, that the spectrum mapping theorem (which would now take the form $e^{\sigma(A)} = \sigma(e^A)\setminus\{0\}$) need not hold and, consequently, spb A and type A are in general different.

Nevertheless, we show that the previous results are true for a special class of generators A, which are roughly those appearing in parabolic partial differential equations. Also we give a wide class of counterexamples, which are not at all pathological, in which $-\infty$ = spb A < type A < $+\infty$.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A SPECTRAL MAPPING THEOREM FOR THE EXPONENTIAL FUNCTION,

AND SOME COUNTEREXAMPLES

Tosio Kato

§0. Introduction

Let A be a linear operator in a complex Banach space X. Consider the mapping $A \to e^A$ and the validity of the spectral mapping theorem

$$\sigma(e^A) = e^{\sigma(A)} \quad , \tag{0.1}$$

where σ denotes the spectrum.

According to the general spectral mapping theorem, (0.1) is true if $A \in B(X)$ (bounded linear operators with domain X). If A is unbounded, (0.1) need note be true even when A is the generator of a strongly continuous group $\{U(t) = e^{tA};\ -\infty < t < \infty\}$. A striking counterexample is given in Hille-Phillips [1, p. 665], in which $U(t)$ is the Riemann-Liouville fractional integration of the imaginary order it on $X = L^p(0,1)$. Here $\sigma(A)$ is empty but $\sigma(e^A)$ is nonempty and is away from zero.

On the other hand, Hille-Phillips [1, p. 460] shows that (0.1) is true (except for zero) if A is the generator of a semigroup $\{e^{tA};\ t > 0\}$ which is norm-continuous for $t > \gamma$ for some constant $\gamma > 0$. The proof in [1] is difficult, however, based on the Gelfand theory of *normed* rings.

The purpose of the present note is twofold. First we give an elementary proof of the Hille-Phillips theorem in the special case when A generates a holomorphic semigroup. We shall then give a wide class of generators A of groups for which $\sigma(A)$ is empty, including the fractional integrals mentioned above as a special case.

§1. A spectral mapping theorem.

We begin with a one-sided inclusion in (0.1) for the generator of a C_0-semigroup.

Theorem 1. Let A be the generator of a semigroup $\{e^{tA};\ t \geq 0\}$ of class C_0. Then $e^{\sigma(A)} \subset \sigma(e^A)$.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

Remark. This is a special case of Corollary 2 to Lemma 16.3.2 of [1, p. 457], in which A may be the generator of any semigroup of class A. But our proof is elementary and short.

Proof. For any complex number z_0, set

$$(1.1) \qquad T = e^{z_0} \int_0^1 e^{t(A-z_0)} dt \in B(X) \quad .$$

Since $(A-z_0)e^{t(A-z_0)}u = (d/dt)e^{t(A-z_0)}u$ for $u \in D(A)$, it follows on integration that

$$(1.2) \qquad T(A-z_0) \subset (A-z_0)T = e^A - e^{z_0} \quad .$$

If $e^{z_0} \in \rho(e^A)$ (ρ denotes the resolvent set), (1.2) gives

$$(1.3) \qquad (A-z_0)^{-1} = (e^A - e^{z_0})^{-1} T \in B(X) \quad ,$$

so that $z_0 \in \rho(A)$. In other words, $z_0 \in \sigma(A)$ implies $e^{z_0} \in \sigma(e^A)$, q.e.d.

Theorem 2. Let A be the generator of a holomorphic C_0-semigroup. Then $e^{\sigma(A)} = \sigma(e^A) \setminus \{0\}$.

Remarks. (a) By a holomorphic C_0-semigroup we mean a semigroup of class C_0 which has an analytic continuation to a sector containing the positive t-axis. It is known (see [1, Theorem 12.8.1]) that A generates such a semigroup if and only if $\rho(A)$ contains a sector

$$(1.4) \qquad \Sigma = \{z \; ; \; |\arg(z-\gamma)| < \omega\} \quad ,$$

where γ is a complex number and $\omega > \pi/2$, and

$$(1.5) \qquad \|(z-A)^{-1}\| \leq M_\varepsilon |z-\gamma|^{-1} \quad \text{for} \quad |\arg(z-\gamma)| \leq \omega - \varepsilon$$

for each $\varepsilon \in (0,\omega)$.

(b) Removing 0 from $\sigma(e^A)$ in Theorem 2 is natural since $e^{\sigma(A)}$ never contains 0 but $\sigma(e^A)$ may well do.

(c) Theorem 2 is a special case of Theorem 16.4.1 of [1].

Proof. In view of Theorem 1, it suffices to show that

$$(1.6) \qquad \sigma(e^A) \setminus \{0\} \subset e^{\sigma(A)} \quad .$$

In the proof we may assume $\gamma = 0$ in (1.4), (1.5) without loss of generality.

To prove (1.6), it suffices in turn to show that

$$0 \neq \zeta_0 \notin e^{\sigma(A)} \quad \text{implies} \quad \zeta_0 \in \rho(e^A) \quad . \tag{1.7}$$

To this end we first note that

$$e^A = \frac{1}{2\pi i} \int_{C_0} e^z (z-A)^{-1} dz \quad , \tag{1.8}$$

where C_0 is a curve in Σ running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, where $\pi/2 < \theta < \omega$ (see e.g. Kato [2, p. 489]).

Given a ζ_0 as in (1.7), consider all the complex numbers z_j $(j = 1,\ldots,m)$ lying to the left of C_0 and satisfying $e^{z_j} = \zeta_0$. Obviously there are at most finitely many such z_j; they are on a vertical line and equally spaced. We may assume that there is no $z \in C_0$ with $e^z = \zeta_0$, by deforming C_0 if necessary.

The assumption in (1.7) implies that $z_j \in \rho(A)$ $(j = 1,\ldots,m)$. Hence we can find a small circle C_j about z_j such that C_j and its interior are in $\rho(A)$. We may assume that C_j, including C_0, are separated from one another.

We now construct a Dunford-type integral

$$S = \frac{1}{2\pi i} \int_C \frac{e^z}{e^z - \zeta_0} (z-A)^{-1} dz \in B(X) \tag{1.9}$$

where

$$C = C_0 + C_1 + \cdots + C_m \quad ,$$

the C_j being assumed to be coherently oriented (so that the C_j with $j \geq 1$ are negatively oriented). Note that the integral (1.9) exists because the integrand is analytic for $z \in C$ and decays exponentially at infinity on C_0.

We note that C_0 in (1.8) may be replaced by C, since there is no contribution to the integral from the C_j, the integrand being analytic on each C_j and its interior. Then we can apply the Dunford integral calculus, to obtain (see e.g. [2, p. 44])

$$e^A S = \frac{1}{2\pi i} \int_C \frac{e^{2z}}{e^z - \zeta_0} (z-A)^{-1} dz \quad . \tag{1.10}$$

Here it should be noted that both e^z and $e^z(e^z-\zeta_0)^{-1}$ are analytic in the closed domain bounded by C. The fact that this domain is unbounded causes no difficulty, since these functions decay rapidly at infinity.

Since

$$\frac{e^{2z}}{e^z-\zeta_0} = \left(1 + \frac{\zeta_0}{e^z-\zeta_0}\right)e^z \quad ,$$

it follows from (1.8) to (1.10) that

$$e^A S = e^A + \zeta_0 S \quad .$$

Hence

$$(\zeta_0-e^A)(1-S) = \zeta_0 \quad .$$

It follows that $\zeta_0 \in \rho(e^A)$, with

$$(\zeta_0 - e^A)^{-1} = \zeta_0^{-1}(1-S) \in B(X) \quad .$$

This proves (1.7), q.e.d.

§2. Counterexamples --- fractional powers of accretive operators.

In this section we show that the counterexample of fractional integrals mentioned in §0 is not an isolated phenomenon.

<u>Theorem</u> 3. Let H be a Hilbert space. Let $B \in B(H)$ be accretive. Then the fractional powers B^α are well defined and form a holomorphic semigroup for $\mathrm{Re}\,\alpha > 0$, with

$$\|B^\alpha\| \le \frac{\sin \pi\xi'}{\pi\xi'(1-\xi')} \|B\|^\xi e^{\pi|\eta|/2} \qquad (\xi = \mathrm{Re}\,\alpha > 0) \tag{2.1}$$

where $\eta = \mathrm{Im}\,\alpha$ and $\xi' = \xi - [\xi]$. If in particular 0 is not an eigenvalue of B, then B^α is strongly continuous for $\mathrm{Re}\,\alpha \geq 0$, and $\{B^{i\eta};\ -\infty < \eta < \infty\}$ is a strongly continuous group with

$$\|B^{i\eta}\| \le e^{\pi|\eta|/2} \quad . \tag{2.2}$$

If, in addition, B is quasi-nilpotent, then the generator iA of the group $\{B^{i\eta}\}$ has empty spectrum, while $e^{\pm iA} = B^{\pm i}$ have nonempty spectra away from 0.

<u>Remark</u>. The estimate (2.2) is sharp; equality holds for $k(x,y) = 1$ (see [1, p. 665]).

Proof. Theorem 3 was proved in Kato [3] except for the last assertion regrding the case when B is quasi-nilpotent. In this case the semigroup $\{B^{\xi};\ \xi > 0\}$ is of type $-\infty$ (i.e. $\lim_{\xi\to+\infty} \xi^{-1}\log\|B^{\xi}\| = -\infty$), so that its generator A has empty spectrum. (Note that $\{B^{\xi}\}$ has generator A if $\{B^{i\eta}\}$ has generator iA.) Since $\{B^{i\eta}\}$ is a group, on the other hand, it is obvious that $e^{\pm iA} = B^{\pm i} \in B(X)$ have nonempty spectra away from 0.

Example. There are abundant examples of operators B satisfying the conditions of Theorem 3. Let $k(x,y)$ be a continuous, hermitian symmetric, nonnegative-definite kernel on $[0,1] \times [0,1]$. $k(x,y)$ defines an integral operator $K \in B(H)$, where $H = L^2(0,1)$, such that $K^* = K \geq 0$. Let B be the associated Volterra operator:

$$Bu(x) = \int_0^x k(x,y)u(y)dy \quad (u \in H) \quad . \tag{2.3}$$

Then B is quasi-nilpoint and accretive, since $2\,\mathrm{Re}(Bu,u) = (Ku,u) \geq 0$. B has no eigenvalue 0 if K is strictly positive, since $Bu = 0$ implies $(Ku,u) = 0$ by the remark above. But B may have no eigenvalue 0 even when K is only semi-definite. The simplest example of B is given by $k(x,y) = 1$. Then B is a simple integration, and $\{B^{\alpha}\}$ is exactly the fractional integrals considered in §0.

§3. The spectral bound and the type.

If A is a closed linear operator in X, we define the spectral bound of A by

$$\mathrm{spb}\ A = \sup \mathrm{Re}\ \sigma(A) = \sup\{\mathrm{Re}\ \lambda;\ \lambda \in \sigma(A)\} \quad . \tag{3.1}$$

We set $\mathrm{spb}\ A = -\infty$ if $\sigma(A)$ is empty.

If A generates a strongly continuous semigroup $\{e^{tA};\ t > 0\}$, the type of A is defined by

$$\mathrm{type}\ A = \lim_{t\to+\infty} t^{-1} \log\|e^{tA}\| = \log\ \mathrm{spr}\ e^{A} < +\infty \quad , \tag{3.2}$$

where spr denotes the spectral radius. (type A is usually referred to the semigroup $\{e^{tA}\}$ rather than to the generator A, but we use the notation type A as a convenient abuse.)

Theorem 1 shows that

$$\text{(3.3)} \qquad \text{spb } A \leq \text{type } A \quad (< +\infty)$$

if A generates a C_0-semigroup. (Actually it is true for more general semigroups.)

Theorem 2 shows that equality holds in (3.3) if A generates a holomorphic semigroup. (Again it is true for more general semigroups.)

Theorem 3 shows that equality in (3.3) need not hold even for the generator A of a strongly continuous group; indeed one has spb $A = -\infty$ while $|\text{type } A| \leq \pi/2$ in Theorem 3.

It may be noted that Greiner-Voigt-Wolff [4] gives examples of positivity-preserving C_0-semigroups $\{e^{tA}\}$ on certain function spaces for which spb $A = -\infty$ and type $A = 0$.

In fact little is known, beyond holomorphic semigroups (or, more generally, norm-continuous semigroups), about the question of when equality holds in (3.3).

REFERENCES

[1] E. Hille and R. S. Phillips, Functional Analysis and semi-groups, Revised Edition, AMS Colloquium Publ. Vol. 31, 1957.

[2] T. Kato, Perturbation theory for linear operators, Second Edition, Springer, 1980.

[3] T. Kato, Fractional powers of dissipative operators, II, J. Math. Soc. Japan 14, 242-248 (1962).

[4] G. Greiner, J. Voigt, and M. Wolff, On the spectral bound of the generator of semigroups of positive operators, preprint, 1981.

TK/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
#2316

2. GOVT ACCESSION NO.
AD-A114 486

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
A Spectral Mapping Theorem for the Exponential Function, and some Counterexamples

5. TYPE OF REPORT & PERIOD COVERED
Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Tosio Kato

8. CONTRACT OR GRANT NUMBER(s)
DAAG29-80-C-0041

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53706

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Work Unit Number 1 - Applied Analysis

11. CONTROLLING OFFICE NAME AND ADDRESS
U. S. Army Research Office
P.O. Box 12211
Research Triangle Park, North Carolina 27709

12. REPORT DATE
January 1982

13. NUMBER OF PAGES
7

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
C_0-semigroups, holomorphic semigroups, spectral mapping theorem, spectral radius, spectral bound, type

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Elementary proofs are given for the (known) theorem that (1) $e^{\sigma(A)} \subset \sigma(e^A)$ if A is the generator of a C_0- semigroup $\{e^{tA}\}$ of linear operators on a Banach space X, and that (2) $e^{\sigma(A)} = \sigma(e^A) \setminus \{0\}$ if $\{e^{tA}\}$ is a holomorphic semigroup. Also a large class of strongly continuous groups $\{e^{tA}\}$ on a Hilbert space H is given such that $\sigma(A)$ is empty. Note that $\sigma(e^A)$ is not empty, and is away from zero, if $\{e^{tA}\}$ is a group. Some related remarks are given on the relationship between the spectral bound of A and the type of $\{e^{tA}\}$.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ME

8